Bootstrap for almost cyclostationary process with jitter effect

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Let $X = \{X(t), t \in \mathbb{R}\}$ be a zero-mean real-valued process that is uniformly almost cyclostationary (UACS) i.e.,

- $E(X^2(s)) < \infty$ for any $s \in \mathbb{R}$,
- autocovariance function $B(t, \tau) = \text{Cov}(X_t, X_{t+\tau}) = E(X(t)X(t + \tau))$ is almost periodic in $t$ uniformly in $\tau$.

Process $X(t)$ is not observed continuously!, but only in time moments $t_k = kh + U_k$, $h > 0$, where r.v. $U_k$ are iid and are independent on $X$.

We observe the discrete time process $X_k = \{X(nh + U_k), n \in \mathbb{Z}\}$. 

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Jitter effect

We assume that each time moment in which the process is observed is disturbed ($t_k = kh + U_k$, $h > 0$) - jitter effect.

When acquiring a signal, the jitter of the sampling clock:

- voluntary (e.g., in the case of compressed sensing to achieve a random signal acquisition, a random jitter on the clock signal can be added)
- involuntary (e.g., the angular acquisitions vibratory signals from rotating machinery. Angular sampling is sensitive to hardware imperfections (optical encoder precision, electrical perturbation, etc. Some of these imperfections can be viewed as non-uniform sampling or a random jitter.)

WE NEED tools to be able to analyze CS/ACS signals in the presence of jitter.
For fully observed process \( X \), Fourier coefficient of the autocovariance function are of the form

\[
a(\lambda, \tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T E(X(t)X(t+\tau)) \exp(-i\lambda s) ds.
\]

Estimator:

\[
\hat{a}_T(\lambda, \tau) = \frac{1}{T} \int_0^T X(t)X(t+\tau) \exp(-i\lambda t) dt
\]

is consistent and asymptotically normal (Hurd and Leśkow (1992), Dehay (1995)).
Problem formulation

**Second-order analysis**

We observe a sample \( \{X(kh + U_k) : 1 \leq k \leq n\} \).

**Problem:** to compute estimates of \( a(\lambda, \tau) \) we need to know values \( X(t)X(t + \tau) \) and \( \tau \) is not always a multiple of \( h \)!

Thus, we need to use some approximation - we approximate \( \tau \) by the nearest multiple of \( h \).

- \( k_{\tau} \) is the nearest integer to \( \tau/h \),
- \( U_{k,\tau} = U_{k+k_{\tau}} + k_{\tau}h - \tau \) is a time perturbation for the time moment \( kh + \tau \).

Estimator of \( a(\lambda, \tau) \) is defined as follows (Dehay and Monsan (2007))

\[
\tilde{a}(\lambda, \tau) = \frac{1}{n} \sum_{k=1}^{n} X(kh + U_k)X((k + k_{\tau})h + U_{k+k_{\tau}}) \exp(-i\lambda kh),
\]

where \( 0 \leq k \leq n \) and \( 0 \leq k + k_{\tau} \leq n \).

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Assumptions

(i) $X$ is sampled at constant rate greater than Nyquist rate with time step $h > 0$ ($h$ is small enough to avoid aliasing);

(ii) the random perturbations $U_k$ are i.i.d. from some distribution on $(-h/2, h/2)$;

(iii) set $\Lambda = \{\lambda \in \mathbb{R} : a(\lambda, \tau) \neq 0$ for some $\tau \in \mathbb{R}\}$ is finite;

(iv) $\sup_t \mathbb{E}|X(t)|^{8+2\eta} < \infty$ for some $\eta > 0$;

(v) $X$ has almost periodic fourth moments, i.e. for each $t \in \mathbb{R}$ $\mathbb{E}\{X(t)^4\} < \infty$ and the function

$(t, \tau_1, \tau_2, \tau_3) \mapsto \mathbb{E}\{X(t)X(t + \tau_1)X(t + \tau_2)X(t + \tau_3)\}$ is almost periodic in $t$ uniformly with respect to $\tau_1, \tau_2, \tau_3 \in \mathbb{R}$;

(vi) $X(t)$ is $\alpha$-mixing and $\sum_{k=1}^{\infty} k^{\alpha} \frac{\eta}{\eta+4} (k) < \infty$;
Properties of $\tilde{a}(\lambda, \tau)$ - Dehay and Monsan (2007)

Theorem 1

Assume that conditions (i) – (iii) and (vi) are fulfilled. Moreover, let
$$\sup_t E|X(t)|^{4+\eta} < \infty \text{ for some } \eta > 0.$$ Then

$$\sqrt{nh} \left( \tilde{a}(\lambda, \tau) - E(\tilde{a}(\lambda, \tau)) \right) \xrightarrow{d} N_2(0, B(\lambda, \tau)).$$

- covariance matrix has very complicated form, depends on unknown parameters;
- to construct confidence intervals for $a(\lambda, \tau)$ we need to resampling approach;
- under the same conditions the multidimensional version of theorem above can be shown.
We observe a sample \((X(h + U_1), X(2h + U_2), \ldots, X(nh + U_n))\). We decompose

\[
\tilde{a}(\lambda, \tau) = \frac{1}{n} \sum_{k=1}^{n} \tilde{b}_k(\tau) \exp(-i\lambda kh),
\]

where

\[
\tilde{b}_k(\tau) = X(kh + U_k)X((k + k\tau)h + U_{k+k\tau}).
\]

\(\tilde{b}_k(\tau), k = 1, 2, \ldots\) is an ACS time series

We apply the Moving Block Bootstrap (Künsch (1989)) to

\[
\left(\left(\tilde{b}_1(\tau), 1\right), \ldots, \left(\tilde{b}_n(\tau), n\right)\right),
\]

where \(\tau\) is fixed.
1. Choose the block length $b < n$. Then $n = lb + r$, $r = 0, \ldots, b - 1$.

2. For $t = 1, 2, \ldots, n - b + 1$ let
   
   $$B_t = \left(\left(\bar{b}_t(\tau), t\right), \ldots, \left(\bar{b}_{t+b-1}(\tau), t + b - 1\right)\right)$$
   
   be a block of the length $b$. From the set $\{B_1, \ldots, B_{n-b+1}\}$ we choose randomly with replacement $l + 1$ blocks $B_1^*, \ldots, B_{l+1}^*$. This means that
   
   $$P^* \left(B_i^* = B_j\right) = \frac{1}{n - b + 1} \quad \text{for} \quad i = 1, \ldots, l + 1, j = 1, \ldots, n - b + 1,$$
   
   where $P^*$ denotes the conditional probability given the sample $\{X(kh + U_k) : 1 \leq k \leq n\}$.

3. Join the selected $l + 1$ blocks $(B_1^*, \ldots, B_{l+1}^*)$ and take the first $n$ observations to get the bootstrap sample
   
   $$\left(\left(\bar{b}_1^*(\tau), 1^*\right), \ldots, \left(\bar{b}_n^*(\tau), n^*\right)\right)$$
   
   of the same length as the original one.
The bootstrap estimator of $a(\lambda, \tau)$ is of the form

$$\tilde{a}^*(\lambda, \tau) = \frac{1}{n} \sum_{k=1}^{n} \tilde{b}_k^*(\tau) \exp(-i\lambda k^* h).$$

**Comment:** $\tilde{b}_k(\tau)$ does not depend on frequency $\lambda$. This allows to calculate the value of $\tilde{a}^*(\lambda, \tau)$ at the same time for many frequencies $\lambda$. Bootstrapping $\tilde{b}_k(\tau) \exp(-i\lambda kh)$ requires repetition of the algorithm for each frequency.
Theorem 2

Under assumptions (i) – (vi) and if $b \to \infty$ as $n \to \infty$ such that $b = o(n)$, we have that

$$
\rho\left( \mathcal{L} \left( \sqrt{nh} (\tilde{a}(\lambda, \tau) - a(\lambda, \tau)) \right), \mathcal{L}^* \left( \sqrt{nh} (\tilde{a}^*(\lambda, \tau) - E^* \tilde{a}^*(\lambda, \tau)) \right) \right) \xrightarrow{p} 0.
$$

By $\rho$ we denote any distance metricizing weak convergence of probability measures on $\mathbb{R}^2$. 

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Let \( r, \tau \in \mathbb{N} \) be fixed and

\[ \lambda = (\lambda_1, \ldots, \lambda_r)' , \]

be vector of frequencies. Moreover, let

\[ \Re a(\lambda, \tau) = (\Re a(\lambda_1, \tau), \ldots, \Re a(\lambda_r, \tau))' , \]
\[ \Im a(\lambda, \tau) = (\Im a(\lambda_1, \tau), \ldots, \Im a(\lambda_r, \tau))' , \]

and

\[ a(\lambda, \tau) = (\Re a(\lambda_1, \tau), \Im a(\lambda_1, \tau), \ldots, \Re a(\lambda_r, \tau), \Im a(\lambda_r, \tau))' . \]

\( \tilde{a}(\lambda, \tau) \) is the estimator of \( a(\lambda, \tau) \) and \( \tilde{a}^*(\lambda, \tau) \) is its bootstrap counterpart.
Theorem 3

Under the assumptions of Theorem 2

$$\rho \left( \mathcal{L} \left\{ \sqrt{nh} (\tilde{a}(\lambda, \tau) - a(\lambda, \tau)) \right\}, \mathcal{L}^* \left\{ \sqrt{nh} (\tilde{a}^* (\lambda, \tau) - E^* \tilde{a}^* (\lambda, \tau)) \right\} \right) \xrightarrow{p} 0,$$

where $\rho$ is a metric metricizing weak convergence in $\mathbb{R}^{2r}$.
Using Theorem 2 one may construct bootstrap pointwise confidence intervals.

In practice for each $\tau$ many frequencies are considered. Thus, the simultaneous confidence intervals are of great importance. To obtain them the consistency of the bootstrap approach for smooth functions of $a(\lambda, \tau)$ needs to be show. This result is a direct consequence of Theorem 3 and the continuous mapping theorem.
Circular Block Bootstrap (CBB) - Politis and Romano (1992)

- the CBB is a modification of the MBB;
- data are treated as wrapped on the circle;
- each observation is present in the same number of blocks;
- we have exactly \( n \) blocks of the length \( b \).
We apply bootstrap to
\[
\left( \left( \bar{b}_1(\tau), 1 \right), \ldots, \left( \bar{b}_n(\tau), n \right) \right),
\]
where \(\tau\) is fixed.

For \(i = 1, 2, \ldots, n - b + 1\)
\[
B_i = \left( \left( \bar{b}_i(\tau), i \right), \ldots, \left( \bar{b}_{i+b-1}(\tau), i + b - 1 \right) \right).
\]

For \(i=n-b+2, \ldots, n\)
\[
B_i = \left( \left( \bar{b}_i(\tau), i \right), \ldots, \left( \bar{b}_n(\tau), n \right), \left( \bar{b}_1(\tau), 1 \right), \ldots, \left( \bar{b}_{b-n+i-1}(\tau), b - n + i - 1 \right) \right).
\]
Circular versions of our bootstrap algorithm:

- The only change in the algorithm is the form of the set of the possible block length choices. Now it is of the form \( \{B_1, \ldots, B_n\} \). All rules how the blocks can be select remain unchanged.

- The bootstrap estimator \( \tilde{a}^*(\lambda, \tau) \) and its multidimensional version \( \tilde{a}^*(\lambda, \tau) \) remain unchanged.

- All consistency results for the circular approach hold under the same assumptions as in the non-circular case.

- The bootstrap estimator of \( a(\lambda, \tau) \) obtained using the circular version of bootstrap algorithm is unbiased i.e., if \( n = lb \)

\[
E^* (\tilde{a}^*(\lambda, \tau)) = \tilde{a}(\lambda, \tau).
\]

This property does not hold for the non-circular bootstrap method.
$X(t) = a(t) \cos(2\pi t/10) + b(t),$

where

- $a(t)$ - stationary zero-mean Gaussian processes with $\sigma = 0.1$,
- $b(t)$ is a correlated additive Gaussian noise

$$b(t) = 0.5b(t - 1) + 0.25b(t - 2) + \epsilon(t),$$

where $\epsilon(t)$ are iid r.v. from $N(0, 0.05)$. 

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\[ \Lambda = \{-0.4\pi, 0, 0.4\pi\}; \]

in each period of length 10 we collect 100 observations;

\[ n = 10000 \text{ (i.e. there are 100 periods)}; \]

95\% bootstrap pointwise and simultaneous confidence intervals for \( \text{Re} (a(\lambda, 0)) \) and \( \text{Im} (a(\lambda, 0)) \);

\[ \lambda \in \{0\text{Hz}, 0.01\text{Hz}, \ldots, 0.5\text{Hz}\}; \]

\[ b \in \{\sqrt[3]{n}, \sqrt{n}\} = \{20, 100\}; \]

\( U_i \) are iid r.v. \( U(-h/2, h/2) \) or \( U_i \) are iid from the truncated normal distribution (\( \mu = 0, \sigma = 0.01 \)).
**Figure:** Results for uniform distribution. Grey color: Pointwise (first row) and simultaneous (second row) confidence intervals for $Re\ (a(\lambda, 0))$ (left column) and $Im\ (a(\lambda, 0))$ (right column), $\lambda \in \{0Hz, 0.01Hz, \ldots, 0.5Hz\}$ and $b = 100$. Black color: estimators of $Re\ (a(\lambda, 0))$ and $Im\ (a(\lambda, 0))$. Nominal coverage probability is 95%.
vibratory signal acquired with a random sampling frequency; the recordings were carried out at CETIM on a gear system with a train of gearing with a ratio of 20/21 functioning continuously until its destruction;
the test was of length 12 days with a daily mechanical appraisal; measurements were collected every 24 h;
the accelerometer signal was acquired with a sampling frequency of 20 KHz; signal was resampled with a uniform random temporal step;
the length of the signal is 30000 and the average of this frequency is 10kHz;
95% bootstrap pointwise and simultaneous confidence intervals for $Re (a(\lambda, 0))$ and $Im (a(\lambda, 0))$;
$\Lambda = \{0 Hz, 340 Hz\}$;
$\lambda \in \{300 Hz, 310 Hz, \ldots, 700 Hz\}$;
$b \in \{\sqrt[3]{n}, \sqrt{n}\} = \{30, 200\}$;
$B = 500$;
Figure: Grey color: Pointwise (first row) and simultaneous (second row) confidence intervals for $\text{Re}(a(\lambda, 0))$ (left column) and $\text{Im}(a(\lambda, 0))$ (right column), $\lambda \in \{300\text{Hz}, 310\text{Hz}, \ldots, 700\text{Hz}\}$ and $b = 200$. Black color: estimators of $\text{Re}(a(\lambda, 0))$ and $\text{Im}(a(\lambda, 0))$. Nominal coverage probability is 95%. 

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